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A Combined Feasible-Infeasible Point Continuation Method for Strongly Monotone Variational Inequality Problems*

JINBAO JIAN
College of Mathematics and Physics, Guangxi University, Nanning City 530004, Guangxi, China
(e-mail: jianjb@gxu.edu.cn)

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Abstract. In this paper, we discuss the variational inequality problems VIP$(X,\ F)$, where F is a strongly monotone function and the convex feasible set X is described by some inequaliy constraints. We present a continuation method for VIP$(X,\ F)$, which solves a sequence of perturbed variational inequality problems PVIP$(X,\ F,\ \varepsilon,\ \mu)$ depending on two parameters $\varepsilon \geqslant 0$ and $\mu > 0$. It is worthy to point out that the method will be a feasible point type when $\varepsilon = 0$ and an infeasible point type when $\varepsilon > 0$, i.e., it is a combined feasible–infeasible point (CFIFP for short) method. We analyse the existence, uniqueness and continuity of the solution to PVIP$(X,\ F,\ \varepsilon,\ \mu)$, and prove that any sequence generated by this method converges to the unique solution of VIP$(X,\ F)$. Moreover, some numerical results of the algorithm are reported which show the algorithm is effective.

Key words: Variational inequality problems, Strongly monotone function, Combined feasible–infeasible point method, Continuation method

1. Introduction

We consider the following variational inequality problem (VIP for short) which is to find a vector $x^* \in X$ such that

$$\text{VIP}(X,\ F)\ \ F(x^*)^T(x - x^*) \geqslant 0, \forall\, x \in X, \tag{1}$$

where $F:\ R^n \to R^n$ is a given function and $X \subseteq R^n$ is called the feasible set of VIP$(X,\ F)$.

It is well known that VIP is a very important research area since it has numerous applications in both mathematics itself and economics. Many social and economic models, convex mathematical programming and so on can be considered as special cases of VIP, the more concrete applications can be seen in [1, 2]. For examples, if X is an open set (example $X = R^n$), VIP$(X,\ F)$ is to find $x^* \in X$ such that $F(x^*) = 0$; if $X = R^n_+ := \{x \in R^n \mid x \geqslant 0\}$, VIP$(X,\ F)$ is equivalent to the

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following nonlinear complementarity problem (NCP for short) of finding a vector $x \in R^n$ such that

$$\text{NCP}(F)\ x \geqslant 0,\ F(x) \geqslant 0,\ x^T F(x) = 0$$

On the other hand, the nonlinear programming problem (NLP) is also closely related to VIP. If $F(x)$ is a gradient function of some real-valued function $f : R^n \to R'$, then VIP$(X,\ F)$ is equivalent to the stationary condition of the optimization problem: $\min\{f(x) \mid x \in X\}$. In this paper, we restrict the feasible set X of VIP$(X,\ F)$ to the following form which arises in many applications:

$$X = \{x \in R^n \mid g_i(x) \geqslant 0,\ i \in I\},\quad I = \{1, \ldots, m\} \tag{2}$$

where $g_i : R^n \to R^1$ $(i \in I)$ are assumed to be concave and continuous differentiable real-valued functions. Harker and Pang [2] showed that VIP$(X,\ F)$ and the following mixed NCP are completely equivalent under the linearly independent constraint qualification:

$$\text{NCP}\ F(x) - \nabla g(x)y = 0,\ g_i(x) \geqslant 0,\ y_i \geqslant 0,\ y_i g_i(x) = 0,\ i \in I, \tag{3}$$

where $\nabla g(x) = (\nabla g_1(x), \ldots, \nabla g_m(x))$ denotes the gradient matrix. Hence one can obtain the solution and properties of VIP$(X,\ F)$ by studying and solving NCP, and the continuation method is a kind of this type, it usually solves VIP (i.e., NCP) throught solving a sequence of perturbed VIP (in short, PVIP) containning some parameters. Kanzow and Jiang [3] and Chen and Harker [4] considered the following PVIP problems denoted here as PVIP$_1(X,\ F,\ \mu)$ and PVIP$_2(X,\ F,\ \varepsilon_1,\ \varepsilon_2,\ \mu)$, respectively, where $(\varepsilon,\ \varepsilon_1,\ \varepsilon_2,\ \mu) > 0$:

$$\text{PVIP}_1(X, F, \mu) \quad \begin{array}{l} F(x) - \nabla g(x)y = 0,\ g_i(x) - z_i = 0, \\ y_i > 0,\ z_i > 0,\ y_i z_i = \mu,\ i \in I. \end{array}$$

$$\text{PVIP}_2(X, F, \varepsilon, \mu) \quad \begin{array}{l} F(x) + \varepsilon_1 x - \nabla g(x)y = 0, \\ g_i(x) + \varepsilon_2 y_i > 0,\ y_i > 0,\ y_i(g_i(x) + \varepsilon_2) = \mu,\ i \in I. \end{array}$$

Obviously, the sequence of solutions of PVIP$_1(X,\ F,\ \mu)$ which approaches the solution of VIP$(X,\ F)$ belongs to the feasible set X, and the method in [3] is a kind of feasible point algorithm. In contrast, Chen and Harker's method [4] generates infeasible points since $\varepsilon_2 > 0$.

In this paper, motivated by the ideas of [3, 4], we introduce a similar PVIP(X, F, ε, μ) which unifies automatically the feasible point and infeasible point continuation methods, i.e., the solution of PVIP(X, F, ε, μ) belongs to the feasible set X for suitable perturbed parameters and is out of X for others. Furthermore, the PVIP(X, F, ε, μ) does not contain the surplus variables z_i used by Kanzow and Jiang. A new continuation method for solving VIP(X, F) is given with the help of PVIP(X, F, ε, μ). We analyse and prove the existence, uniqueness, continuity and convergence of the solution to PVIP(X, F, ε, μ).

Throughout this paper, all vectors are column vectors, the n-dimensional real vector space is denoted by R^n, the vector $(x^T, y^T)^T$ is usually abbreviated as (x, y).

2. The CFIFP continuation method

Throughout this paper, we assume the function F is continuously differentiable and $g_i(i \in I)$ are concave and twice continuously differentiable. We first recall some definitions and results which are well known.

DEFINITION 1. Function $F : R^n \to R^n$ is said to be strongly monotone (with modulus α) over a set $X \subset R^n$ if there exists an $\alpha > 0$ such that

$$\big(F(x_1) - F(x_2)\big)^T (x_1 - x_2) \geqslant \alpha \|x_1 - x_2\|^2, \forall x_1, x_2 \in X. \tag{4}$$

If $X = R^n$, we call F strongly monotone for short. F is assumed to be strongly monotone with modulus α throughout this paper.

THEOREM 1 [2]. *If X is a closed, convex, and nonempty subset of R^n, let function $F : X \to R^n$ be strongly monotone over X. Then VIP(X, F) has a unique solution.*

DEFINITION 2. A vector $x \in X = \{x \in R^n \mid g(x) \geqslant 0\}$ is said to satisfy the linear independence constraint qualification (in short, LICQ), or call the LICQ holds at x, if the gradient vectors $\{\nabla g_i(x) : g_i(x) = 0, i \in I\}$ are linearly independent.

We consider the following perturbed variational inequality problem which our continuation method is based on, denoted by PVIP(X, F, ε, μ) :

$$F(x) - \nabla g(x) y = 0 \tag{5}$$

$$g_i(x) + \varepsilon y_i > 0, \; y_i > 0, \; \big(g_i(x) + \varepsilon y_i\big) y_i = \mu, \; i \in I \tag{6}$$

with perturbation parameters $\varepsilon \geqslant 0$, $\mu > 0$.

REMARK 1. Although the perturbed formulas (6) have the same form as (9) used by Chen and Harker [4], there are essential differences between them. The parameter ε in (6) above is allowed to be zero throughout this paper, and it must be a strictly positive number in [4].

We can easily reformulate $(g_i(x) + \varepsilon y_i) y_i = \mu$ as follows.

$$(g_i(x) + \varepsilon y_i)^2 + y_i^2 + 4(g_i(x) + \varepsilon y_i) y_i = 4\mu + (g_i(x) + \varepsilon y_i)^2 + y_i^2$$

$$(g_i(x) + \varepsilon y_i + y_i)^2 = 4\mu + \big((1 - \varepsilon) y_i - g_i(x)\big)^2$$

Notice for $y_i > 0$, $(g_i(x) + \varepsilon y_i) > 0$, one obtains

$$J_i(x, y, \varepsilon, \mu) := (1 + \varepsilon) y_i + g_i(x) - \sqrt{4\mu + \big((1 - \varepsilon) y_i - g_i(x)\big)^2} = 0$$

Let $J(x, y, \varepsilon, \mu) = \big(J_i(x, y, \varepsilon, \mu), i \in I\big)$, then one has

LEMMA 1. *For any parameters $\varepsilon \in R^1$ and $\mu > 0$, equations $J(x, y, \varepsilon, \mu) = 0$ and formula (6) are completely equivalent.*

Proof. Suppose (6) holds, in view of above transformation, one can directly gets $J(x, y, \varepsilon, \mu) = 0$. Conversely, if $J(x, y, \varepsilon, \mu) = 0$, then

$$(1+\varepsilon)y_i + g_i(x) = \sqrt{4\mu + \big((1-\varepsilon)y_i - g_i(x)\big)^2} \geqslant \sqrt{4\mu} > 0, i \in I$$

Furthermore, using the square of above equations, we have $(g_i(x) + \varepsilon y_i)y_i = \mu > 0, i \in I$. Combining this with $(g_i(x) + \varepsilon y_i) + y_i > 0 (i \in I)$, one obtains $g_i(x) + \varepsilon y_i > 0, y_i > 0 (i \in I)$. Hence formulas (6) hold. □

From Lemma 1, one has directly the following result

THEOREM 2. *Point (x, y) is a solution of PVIP(X, F, ε, μ) if and only if it is a solution of the following system of nonlinear equations, denoted by $\Phi(x, y, \varepsilon, \mu) = 0$:*

$$F(x) - \nabla g(x) y = 0 \tag{7}$$

$$(1+\varepsilon)y_i + g_i(x) - \sqrt{4\mu + \big((1-\varepsilon)y_i - g_i(x)\big)^2} = 0, \quad i \in I \tag{8}$$

As some more effective methods, such as Newton's method and so on, to be used to solve $\Phi(x, y, \varepsilon, \mu) = 0$, we must discuss the properties of the Jacobian matrix of $\Phi(x, y, \varepsilon, \mu)$. We first give the following Lemma which will be used in the proof of Theorem 3.

LEMMA 2. *Let $H \in R^{n\times n}$ be a positive definite matrix, $R, D \in R^{m\times m}$ be positive definite and diagmal matrices, and let $A \in R^{n\times m}$ be an arbitrary matrix. Then the following matrix M is nonsingular*

$$M := \begin{pmatrix} H & AR \\ -A^T & D \end{pmatrix}$$

Proof. Let $w = (x, y) \in R^n \times R^m$, and $Mw = 0$. Then $Hx + ARy = 0, -A^T x + Dy = 0$. So $x^T Hx = -x^T ARy, x^T A = y^T D^T$. Using the assumed conditions and this relations, one has: $0 \leqslant x^T Hx = -y^T D^T Ry \leqslant 0$, which shows $x = 0, y = 0$. Thus $Mw = 0$ implies $w = 0$, that is M is nonsingular. □

THEOREM 3. *The Jacobian matrices $\nabla\Phi(x, y, \varepsilon, \mu)^T$ of $\Phi(x, y, \varepsilon, \mu)$ are nonsingular for all $(x, y) \in R^n \times R^m_+$ and all $\varepsilon \geqslant 0, \mu > 0$. If the functions $g_i(x) (i \in I)$ are affine-linear, then $\nabla\Phi(x, y, \varepsilon, \mu)$ is nonsingular for all $(x, y) \in R^n \times R^m$ and all $\varepsilon \geqslant 0, \mu > 0$.*

Proof. For $i \in I$, denote

$$d_i := d_i(x, y, \varepsilon, \mu) = (1+\varepsilon) - \frac{(1-\varepsilon)y_i - g_i(x)}{\sqrt{4\mu + \big((1-\varepsilon)y_i - g_i(x)\big)^2}}(1-\varepsilon),$$

$$r_i := r_i(x, y, \varepsilon, \mu) = 1 + \frac{(1-\varepsilon)y_i - g_i(x)}{\sqrt{4\mu + \big((1-\varepsilon)y_i - g_i(x)\big)^2}},$$

$$L(x, y) = F(x) - \nabla g(x)y,$$

$$R := R(x, y, \varepsilon, \mu) = \text{diag}\big(r_i, i \in I\big), D := D(x, y, \varepsilon, \mu) = \text{diag}\big(d_i, i \in I\big).$$

Then the Jacobian matrices of $J(x, y, \varepsilon, \mu)$ and $\Phi(x, y, \varepsilon, \mu)$ can be expressed as follows:

$$\nabla_x J(x, y, \varepsilon, \mu)^T = (\nabla g(x)R)^T, \nabla_y J(x, y, \varepsilon, \mu)^T = D,$$

$$\nabla\Phi(x, y, \varepsilon, \mu)^T = \begin{pmatrix} \nabla_x L(x, y) & \nabla g(x)R \\ -\nabla g(x)^T & D \end{pmatrix}^T.$$

Since F is strongly monotone and $g_i(i \in I)$ are concave, $\nabla F(x)$ is positive definite and $-\nabla^2 g_i(x)(i \in I)$ are positive semidefinite. So $H := \nabla_x L(x, y) = \nabla F(x) - \sum_{i\in I} y_i \nabla^2 g_i(x)$ is positive definite for all $(x, y) \in R^n \times R^m_+$. On the other hand, in view of

$$\gamma_i := \frac{|(1-\varepsilon)y_i - g_i(x)|}{\sqrt{4\mu + \big((1-\varepsilon)y_i - g_i(x)\big)^2}} < 1,$$

we know $r_i \geqslant 1-\gamma_i > 0, i \in I$. Thus R is positive definite for all $\varepsilon \geqslant 0, \mu > 0$ and all $(x, y) \in R^n \times R^m$. For matrix D, if $\varepsilon = 1$, then $d_i = 2 > 0$; if $\varepsilon \neq 1$ $(1-\varepsilon \neq 0)$, then $d_i > 1 + \varepsilon - |1-\varepsilon| \geqslant \min\{2\varepsilon, 2\} \geqslant 0$. So D is positive definite too. Hence the Jacobian $\nabla\Phi(x, y, \varepsilon, \mu)^T$ is nonsingular for all $(x, y) \in R^n \times R^m$ and all $\varepsilon \geqslant 0, \mu > 0$ since matrices $H, R, D, A := \nabla g(x)$ satisfy the conditions in Lemma 2. □

To conclude this section, we present the following continuation algorithm for VIP(X, F) (i.e., for NCP) based on the system of equations $\Phi(x, y, \varepsilon, \mu) = 0$ given by (7) and (8).

Initiation Step. Choose a stopping tolerance $\delta > 0$ and an error function err(x, y) (its specific construction can be seen from (17) in this paper or in [4]), choose any initial point $(x_o, y_o) \in R^n \times R^m$ and any sequences $\{\varepsilon_k\}, \{\mu_k\}$ such that

$$\varepsilon_k \geqslant 0, \lim_{k\to\infty} \varepsilon_k = 0; \mu_k > 0, \lim_{k\to\infty} \mu_k = 0.$$

Let $k = 0$ and go to Main Step.

Main Step.

(1) Starting with (x_k, y_k), solve equation $\Phi(x, y, \varepsilon_k, \mu_k) = 0$ approximately for (x_{k+1}, y_{k+1}).

(2) If $\mathrm{err}(x_{k+1}, y_{k+1}) < \delta$, stop. Otherwise, let $k := k + 1$, go to Step (1).

THEOREM 4 [6]. *Let (x^*, y^*) be a solution of $\Phi(x, y, \varepsilon, \mu) = 0$, i.e., it is a solution of* (7) *and* (8). *Suppose one use Newton's method to solve equation system* (7)–(8) *with the initial point (x_o, y_o) located in a small neighborhood of (x^*, y^*). Then this method will converge (x^*, y^*) at a quadratic rate.*

3. Existence and uniqueness of a solution to PVIP $(\mathbf{X}, \mathbf{F}, \varepsilon, \mu)$

In this section, we will prove PVIP(X, F, ε, μ) has a unique solution which is continuous in the parameter μ for all $\varepsilon \geqslant 0$, $\mu > 0$. We first discuss uniqueness and continuity.

LEMMA 3. *Let $\mu_1 > 0$, $\mu_2 > 0$, (x_1, y_1) and (x_2, y_2) are solutions of PVIP$(X, F, \varepsilon, \mu_1)$ and PVIP$(X, F, \varepsilon, \mu_2)$, respectively, then*

$$\alpha\|x_1 - x_2\|^2 + \varepsilon\|y_1 - y_2\|^2 \leqslant m|\mu_1 - \mu_2| \tag{9}$$

Proof. Since (x_1, y_1) and (x_2, y_2) are solutions of PVIP$(X, F, \varepsilon, \mu_1)$ and PVIP$(X, F, \varepsilon, \mu_2)$, respectively, we have from (5) and (6)

$$F(x_1)^T(x_1 - x_2) - (\nabla g(x_1)y_1)^T(x_1 - x_2) = 0,$$
$$F(x_2)^T(x_2 - x_1) - (\nabla g(x_2)y_2)^T(x_2 - x_1) = 0.$$

Adding these two equalities, we have

$$\big(F(x_1) - F(x_2)\big)^T(x_1 - x_2) = y_1^T\nabla g(x_1)^T(x_1 - x_2) + y_2^T\nabla g(x_2)^T(x_2 - x_1). \tag{10}$$

Using the strong monotonicity of F and the concavity of $g_i(i \in I)$, and noting $y_1 > 0$, $y_2 > 0$, one has

$$\big(F(x_1) - F(x_2)\big)^T(x_1 - x_2) \geqslant \alpha\|x_1 - x_2\|^2,$$
$$y_1^T\nabla g(x_1)^T(x_1 - x_2) \leqslant y_1^T\big(g(x_1) - g(x_2)\big),$$
$$y_2^T\nabla g(x_2)^T(x_2 - x_1) \leqslant y_2^T\big(g(x_2) - g(x_1)\big).$$

Substituting the above inequalities into (10), we obtain

$$\alpha\|x_1 - x_2\|^2 \leqslant (y_1 - y_2)^T(g(x_1) - g(x_2)). \tag{11}$$

Since $g_i(x_1) + \varepsilon y_{1i} = \frac{\mu_1}{y_{1i}}$, $g_i(x_2) + \varepsilon y_{2i} = \frac{\mu_2}{y_{2i}}$, $i \in I$, we have

$$(y_1 - y_2)^T(g(x_1) - g(x_2)) = -\varepsilon\|y_1 - y_2\|^2 + \sum_{i\in I} \frac{(y_{1i} - y_{2i})(\mu_1 y_{2i} - \mu_2 y_{1i})}{y_{1i} y_{2i}}.$$

In view of $y_k > 0$, $\mu_k > 0$, using Lemma 3.9 in [4], we get

$$(y_{1i} - y_{2i})(\mu_1 y_{2i} - \mu_2 y_{1i}) \leqslant |\mu_1 y_{1i} y_{2i} - \mu_2 y_{1i} y_{2i}| = y_{1i} y_{2i}|\mu_1 - \mu_2|.$$

Hence

$$(y_1 - y_2)^T(g(x_1) - g(x_2)) \leqslant -\varepsilon\|y_1 - y_2\|^2 + m|\mu_1 - \mu_2|.$$

Thus conclude (9) holds from this inequality and (11). □

THEOREM 5. *PVIP(X, F, ε, μ) has at most one solution for all $\varepsilon \geqslant 0$, $\mu > 0$. Furthermore, the solution of PVIP(X, F, ε, μ) is continuous in the parameter μ.*

Proof. Let (x_1, y_1), (x_2, y_2) be two solutions of PVIP(X, F, ε, μ). Let $\mu_1 = \mu_2 = \mu$, and use (9), we have $\alpha\|x_1 - x_2\|^2 + \varepsilon\|y_1 - y_2\|^2 \leqslant 0$, so $x_1 = x_2$, $\varepsilon y_1 = \varepsilon y_2$. Obviously, $y_1 = y_2$ if $\varepsilon > 0$. If $\varepsilon = 0$, using $(g_i(x_1) + \varepsilon y_{1i})y_{1i} = \mu = (g_i(x_2) + \varepsilon y_{2i})y_{2i} > 0$, and $x_1 = x_2$, one gets directly $y_1 = y_2$. Thus $(x_1, y_1) = (x_2, y_2)$. The continuity follows directly from (9). □

In the remainder part of this section, we will show PVIP(X, F, ε, μ) has a unique solution for all parameters $\varepsilon \geqslant 0$, $\mu > 0$. Our main technique is to transform equivalently PVIP(X, F, ε, μ) into some variational inequality problem which is known to have a solution. For $\varepsilon \geqslant 0$, define $\widetilde{F} : R^n \times R^m \to R^n \times R^m$, $\widetilde{g}_i : R^n \times R^m \to R^1$, $i \in I$, by

$$\widetilde{F}(x, z) = \big(F(x), z_1, ..., z_m\big),\ \widetilde{g}_i(x, z) = g_i(x) + \sqrt{\varepsilon} z_i,\ i \in I. \tag{12}$$

Let $\widetilde{X} = \{(x, z) | \widetilde{g}(x, z) \geqslant 0\}$, we consider the following problem, denoted by PVIP$(\widetilde{X}, \widetilde{F}, \mu)$:

$$\widetilde{F}(x, z) - \nabla\widetilde{g}(x, z)y = 0, \tag{13}$$

$$\widetilde{g}_i(x, z) > 0,\ y_i > 0,\ y_i\widetilde{g}_i(x, z) = \mu,\ i \in I. \tag{14}$$

LEMMA 4. *Vector (x, z, y) is a solution of PVIP$(\widetilde{X}, \widetilde{F}, \mu)$ if and only if (x, y) solves PVIP(X, F, ε, μ) and $z = \sqrt{\varepsilon} y$.*

Proof. Let $e_i \in R^m$ be the vectors whose ith component is one and all others are zero. Since $\nabla\widetilde{g}_i(x, z) = \big(\nabla g_i(x), \sqrt{\varepsilon} e_i\big)(i \in I)$, one knows (13) is equivalent to

$$F(x) - \nabla g(x)y = 0,\ z - \sqrt{\varepsilon} y = 0. \tag{15}$$

If (x, z, y) satisfies (13) and (14), then (15) holds. Moreover, (5) holds, $z_i = \sqrt{\varepsilon} y_i$, $y_i > 0$ and

$$\widetilde{g}_i(x, z) = g_i(x) + \sqrt{\varepsilon} z_i = g_i(x) + \varepsilon y_i > 0,$$
$$y_i \widetilde{g}_i(x, z) = (g_i(x) + \varepsilon y_i) y_i = \mu, i \in I.$$

Hence (x, y) solves (5) and (6), i.e., PVIP(X, F, ε, μ).

Conversely, suppose (x, y) solves PVIP(X, F, ε, μ), $z = \sqrt{\varepsilon} y$, then (x, z, y) satisfies (15), i.e., (13) holds for (x, z, y). One has from (6)

$$y_i(g_i(x) + \varepsilon y_i) = y_i(g_i(x) + \sqrt{\varepsilon} z_i) = y_i \widetilde{g}_i(x, z) = \mu;\ y_i > 0,$$

$$g_i(x) + \varepsilon y_i = g_i(x) + \sqrt{\varepsilon} z_i = \widetilde{g}_i(x, z) > 0.$$

Hence (x, z, y) satisfies (14). Moreover, it solves PVIP$(\widetilde{X}, \widetilde{F}, \mu)$. □

LEMMA 5. *Let* $\varepsilon \geqslant 0$.

(i) *If* $F(x)$ *is strongly monotone over* R^n*, then* $\widetilde{F}(x, z)$ *is strongly monotone over* $R^n \times R^m$.

(ii) *If* $g_i(x)$ *is concave over* R^n*, then* $\widetilde{g}_i(x, z)$ *is concave over* $R^n \times R^m$.

(iii) *If LICQ holds at the solution of VIP*(X, F)*, then LICQ holds at the solution of VIP*$(\widetilde{X}, \widetilde{F})$.

Proof. Parts (i) and (ii) are obvious. We discuss part (iii), first of all if $\varepsilon > 0$, then the gradient vectors $\nabla \widetilde{g}_i(x, z) (i \in I)$ are linear independent for all $(x, z) \in R^n \times R^m$. If $\varepsilon = 0$, it is known that any solution of VIP$(\widetilde{X}, \widetilde{F})$ must be a solution of VIP(X, F). Hence LICQ of $\widetilde{X}$ holds since LICQ holds and $\nabla \widetilde{g}_i(x, z) = (\nabla g_i(x), 0)$. □

THEOREM 6. *Suppose* F *is strongly monotone, and LICQ holds at the solution of VIP*(X, F)*. Then the perturbed variational inequality problems PVIP*(X, F, ε, μ) *have a unique solution for all* $\varepsilon \geqslant 0, \mu > 0$.

Proof. In view of Theorem 5, it is sufficient to show the existence part. Since VIP$(\widetilde{X}, \widetilde{F})$ satisfies all assumptions of Theorem 3.12 in [3] by Lemma 5, using Theorem 3.12 in [3], we can conclude that the associated perturbed variational inequality problems PVIP$(\widetilde{X}, \widetilde{F}, \mu)$ have a solution for all $\varepsilon \geqslant 0, \mu > 0$. Thus problems PVIP$(X, F, \varepsilon, \mu)$ have a solution for all $\varepsilon \geqslant 0, \mu > 0$ from Lemma 4. □

4. The convergence of the method

In the previous section, we have discussed the existence, uniqueness and continuity of the solution of PVIP(X, F, ε, μ). In this section, we will prove the solution (x_k, y_k) of PVIP$(X, F, \varepsilon_k, \mu_k)$ approaches the unique solution (x^*, y^*) of VIP(X, F). Let

$$R_+ = \{s \in R^1 | s \geqslant 0\}, R_{++} = \{s \in R^1 | s > 0\}.$$

LEMMA 6. *Let* $w_{\varepsilon\mu} = (x_{\varepsilon\mu}, y_{\varepsilon\mu})$ *be the unique soluction of PVIP* (X, F, ε, μ) *for* $\varepsilon \geqslant 0, \mu > 0$. *If a set* $\Omega \subset R_+ \times R_{++}$ *is bounded, then* $\{w_{\varepsilon\mu} | (\varepsilon, \mu) \in \Omega\}$ *is bounded.*

Proof. Suppose there exists $\eta > 0$ such that $\varepsilon \leqslant \eta, \mu \leqslant \eta, (\varepsilon, \mu) \in \Omega$. Let $\overline{\mu} > 0$ be a fixed parameter. From Theorem 6, one knows PVIP$(X, F, 0, \overline{\mu})$ has a unique solution $\overline{w} = (\overline{x}, \overline{y})$. Moreover, we have from (11)

$$\alpha\|x_{\varepsilon\mu} - \overline{x}\|^2 \leqslant (y_{\varepsilon\mu} - \overline{y})^T(g(x_{\varepsilon\mu}) - g(\overline{x})),$$

$$(y_{\varepsilon\mu} - \overline{y})^T(g(x_{\varepsilon\mu}) - g(\overline{x})) = (y_{\varepsilon\mu} - \overline{y})^T g(x_{\varepsilon\mu}) - y_{\varepsilon\mu}^T g(\overline{x}) + \overline{y}^T g(\overline{x}).$$

Since

$$\overline{y} > 0, g(x_{\varepsilon\mu}) + \varepsilon y_{\varepsilon\mu} > 0, y_{\varepsilon\mu}^T(g(x_{\varepsilon\mu}) + \varepsilon y_{\varepsilon\mu}) = m\mu,$$

we have

$$\begin{aligned}(y_{\varepsilon\mu} - \overline{y})^T g(x_{\varepsilon\mu}) = &- \varepsilon(y_{\varepsilon\mu} - \overline{y})^T y_{\varepsilon\mu} + y_{\varepsilon\mu}^T(g(x_{\varepsilon\mu}) + \varepsilon y_{\varepsilon\mu}) \\ &- \overline{y}^T(g(x_{\varepsilon\mu}) + \varepsilon y_{\varepsilon\mu}) \\ \leqslant &- \varepsilon(y_{\varepsilon\mu} - \overline{y})^T y_{\varepsilon\mu} + m\mu.\end{aligned}$$

In view of $g(\overline{x}) > 0$, $y_{\varepsilon\mu} > 0$, $\overline{y}^T g(\overline{x}) = m\overline{\mu}$, we obtain : $\alpha\|x_{\varepsilon\mu} - \overline{x}\|^2 \leqslant -\varepsilon(y_{\varepsilon\mu} - \overline{y})^T y_{\varepsilon\mu} + m(\mu + \overline{\mu})$. So

$$\alpha\|x_{\varepsilon\mu} - \overline{x}\|^2 + \varepsilon(y_{\varepsilon\mu} - \overline{y})^T y_{\varepsilon\mu} \leqslant m(\mu + \overline{\mu}). \tag{16}$$

Using (16) and $(y_{\varepsilon\mu} - \overline{y})^T y_{\varepsilon\mu} = \|y_{\varepsilon\mu} - \frac{1}{2}\overline{y}\|^2 - \frac{1}{4}\|\overline{y}\|^2$, we have

$$\alpha\|x_{\varepsilon\mu} - \overline{x}\|^2 + \varepsilon\|y_{\varepsilon\mu} - \frac{1}{2}\overline{y}\|^2 \leqslant m(\mu + \overline{\mu}) + \frac{1}{4}\varepsilon\|\overline{y}\|^2,$$

$$\|x_{\varepsilon\mu} - \overline{x}\|^2 \leqslant \alpha^{-1}(m\eta + m\overline{\mu} + \frac{1}{2}\eta\|\overline{y}\|^2),$$

which guarantees $\{x_{\varepsilon\mu} | (\varepsilon, \mu) \in \Omega\}$ is bounded. On the other hand, using (11) again, we have

$$\begin{aligned}0 \leqslant \alpha\|x_{\varepsilon\mu} - \overline{x}\|^2 &\leqslant (y_{\varepsilon\mu} - \overline{y})^T(g(x_{\varepsilon\mu}) - g(\overline{x})) \\ &= y_{\varepsilon\mu}^T g(x_{\varepsilon\mu}) - y_{\varepsilon\mu}^T g(\overline{x}) - \overline{y}^T g(x_{\varepsilon\mu}) + \overline{y}^T g(\overline{x}) \\ &= m\mu - \varepsilon\|y_{\varepsilon\mu}\|^2 - y_{\varepsilon\mu}^T g(\overline{x}) - \overline{y}^T g(x_{\varepsilon\mu}) + m\overline{\mu} \\ &\leqslant m\mu + m\overline{\mu} - y_{\varepsilon\mu}^T g(\overline{x}) - \overline{y}^T g(x_{\varepsilon\mu}).\end{aligned}$$

Hence

$$y_{\varepsilon\mu}^T g(\overline{x}) \leqslant m\mu + m\overline{\mu} - \overline{y}^T g(x_{\varepsilon\mu})$$

Since $\{x_{\varepsilon\mu} | (\varepsilon, \mu) \in \Omega\}$ is bounded and $g(.)$ is continuous, there exists a constant $M > 0$ such that $-\overline{y}^T g(x_{\varepsilon\mu}) \leqslant M$. Thus $y_{\varepsilon\mu}^T g(\overline{x}) \leqslant m\eta + m\overline{\mu} + M$, This inequality and $y_{\varepsilon\mu} > 0$, $g(\overline{x}) > 0$ conclude $\{y_{\varepsilon\mu} | (\varepsilon, \mu) \in \Omega\}$ is bounded. So we have finished the proof of Lemma 6. □

THEOREM 7. *If the parameters* $\{(\varepsilon_k, \mu_k)\}$ *chosen in the continuation method are arbitrary such that* $\varepsilon_k \geqslant 0, \mu_k > 0, (\varepsilon_k, \mu_k) \to (0, 0), k \to \infty$. *Then the entire sequence* $\{w_k = (x_k, y_k)\}$ *generated by the continuation method converges to the unique solution* (x^*, y^*) *of VIP*(X, F).

Proof. From Lemma 6 and $(\varepsilon_k, \mu_k) \to (0, 0)$, we know $\{w_k\}$ is bounded, so it has at least one accumulation point $(\hat{x}, \hat{y})$. Since (x_k, y_k) satisfies (5) and (6), one has

$$F(x_k) - \nabla g(x_k) y_k = 0,$$

$$g_i(x_k) + \varepsilon_k y_{ki} > 0,\ y_{ki} > 0,\ (g_i(x_k) + \varepsilon_k y_{ki}) y_{ki} = \mu_k,\ i \in I.$$

In view of $(x_k, y_k, \varepsilon_k, \mu_k) \to (\hat{x}, \hat{y}, 0, 0), k \in K$, and taking limit in above equalities $(k \in K)$, one has

$$F(\hat{x}) + \nabla g(\hat{y})\hat{y} = 0,\ g_i(\hat{x}) \geqslant 0,\ \hat{y}_i \geqslant 0,\ \hat{y}_i g_i(\hat{x}) = 0,\ i \in I.$$

Thus $(\hat{x}, \hat{y})$ is a solution of VIP(X, F). Moreover, $(\hat{x}, \hat{y}) = (x^*, y^*)$ since the solution of VIP(X, F) is unique.

So we have shown that $\{w_k = (x_k, y_k)\}$ has a unique accumulation point (x^*, y^*). Thus the entire sequence $\{(x_k, y_k)\}$ converges to the solution (x^*, y^*) of VIP(X, F) by the boundedness of $\{(x_k, y_k)\}$. □

5. Numerical results

In this section, two small examples have been solved by the proposed algorithm, and the numerical results have shown that the proposed algorithm is effective. The error function is chosen as

$$err(x, y) = \|F(x) - \sum_{i \in I} y_i \nabla g_i(x)\|^2 + \sum_{i \in I} (\min\{g_i(x), y_i\})^2. \tag{17}$$

The parameter sequences $\{\varepsilon_k\}$ and $\{\mu_k\}$ are chosen in three cases for a same problem as follows:

$$\varepsilon_k \equiv 0,\ \mu_k = (0.5)^{k+1},\ k = 0, 1, 2, \cdots, \tag{18}$$

$$\varepsilon_o = 0.5,\ \varepsilon_1 = (0.5)^2,\ \varepsilon_k = \begin{cases} 0, & k \geqslant 2 \text{ and } k \text{ is even}, \\ (0.5)^{\frac{k+3}{2}}, & k \geqslant 3 \text{ and } k \text{ is odd}, \end{cases} \quad \mu_k = (0.5)^{k+1}, \tag{19}$$

$$\varepsilon_k = \mu_k = (0.5)^{k+1}, k = 0, 1, 2, \cdots , \tag{20}$$

and the vector y_k is omitted in Tables 1–6.

As the parameters are chosen as (18), all iteration points $x_k (k > 0)$ are feasible (see Tables 1, 4). As they are chosen as (19), some iteration points are feasible, but some are infeasible (see Tables 2, 5), i.e., the method is a combined feasible–infeasible algorithm. When the parameters are chosen as (20), most iteration points are infeasible (see Tables 3, 6).

EXAMPLE 1. This problem is taken from [7]. Let

$$F(x) = \begin{pmatrix} 2x_1 + 0.2x_1^3 - 0.5x_2 + 0.1x_3 - 4 \\ -0.5x_1 + x_2 + 0.1x_2^3 + 0.5 \\ 0.5x_1 - 0.2x_2 + 2x_3 - 0.5 \end{pmatrix},$$

$$X = \{x \in R^3 \mid g(x) = -x_1^2 - 0.4x_2^2 - 0.6x_3^2 + 1 \geqslant 0\}.$$

It is easily verified that Example 1 has the solution $x^* = (1.0, 0.0, 0.0)^T$. The computational results for the staring point $(x_o; y_o) = (100.0, 10.0, 10.0; 1.0)^T$ are shown in Tables 1,2,3, where the parameters sequences $\{\varepsilon_k\}$ and $\{\mu_k\}$ are chosen as (18), (19) and (20), respectively .

EXAMPLE 2. *Let*

$$F(x) = \begin{pmatrix} 3.0 & -4.0 & -16.0 \\ 4.0 & 1.0 & -5.0 \\ 16.0 & 5.0 & 2.0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

$$X = \{x \in R^3 \mid g_4(x) = x_1 + x_2 + x_3 - 10 \geqslant 0, g_i(x) = x_i \geqslant 0, i = 1, 2, 3.\}.$$

Example 2 is a modification of examples used in [8, 9]. The exact solution for Example 2 is $x^* = (10.0, 0.0, 0.0)^T$. The numerical results for solving Example 2 with initial point $(x_o; y_o) = (0.0, 12.0, 10.0; 1.0, 1.0, 1.0, 1.0)^T$ are shown in Tables 4–6, where the parameter sequences $\{\varepsilon_k\}$, $\{\mu_k\}$ are chosen as (18), (19) and (20), respectively.

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Table 1. Result for Example 1: parameters are chosen as (18)

Iteration k	x_1	x_2	x_3	$g(x)$	$err(x, y)$
0	100.00000	10.0000000	10.0000000	-10099.0	101989801.00
1	0.8020101	−0.0616438	0.0234837	0.35493	0.1259745456
2	0.8854567	−0.0536393	0.0137003	0.21470	0.0460973285
3	0.9370589	−0.0481093	0.0067434	0.12097	0.0146331537
4	0.9666197	−0.0448084	0.0024483	0.06484	0.0042041891
5	0.9826396	−0.0429830	0.0000268	0.03368	0.0011343702
6	0.9910151	−0.0420198	−0.0012653	0.01718	0.0002952209
7	0.9953029	−0.0415246	−0.0019339	0.00868	0.0000753478
8	0.9974731	−0.0412735	−0.0022740	0.00436	0.0000190380
9	0.9985650	−0.0411472	−0.0024456	0.00219	0.0000045875
10	0.9991126	−0.0410839	−0.0025318	0.00109	0.0000012029
11	0.9993868	−0.0410522	−0.0025750	0.00055	0.0000003040

Table 2. Result for Example 1: parameters are chosen as (19)

Iteration k	x_1	x_2	x_3	$g(x)$	$err(x, y)$
0	100.00000	10.0000000	10.0000000	−10099.0	101989801.00
1	0.9731280	−0.0441109	0.0014698	0.05224	0.0027292029
2	0.9814327	−0.0431786	0.0002079	0.03604	0.0012991802
3	0.9370581	−0.0481340	0.0067420	0.12097	0.0146332864
4	1.0164387	−0.0390526	−0.0052974	−0.03377	0.0011407263
5	0.9826398	−0.0429756	0.0000273	0.03368	0.0011343687
6	1.0170676	−0.0389826	−0.0053995	−0.03505	0.0012286349
7	0.9953031	−0.0415170	−0.0019334	0.00868	0.0000753487
8	1.0109067	−0.0397071	−0.0044062	−0.02257	0.0005096235
9	0.9985651	−0.0411395	−0.0024451	0.00219	0.0000047883
10	1.0059498	−0.0402874	−0.0036141	−0.01259	0.0001585655
11	0.9993870	−0.0410445	−0.0025745	0.00055	0.00000003048

Table 3. Result for Example 1: parameters are chosen as (20)

Iteration k	x_1	x_2	x_3	$g(x)$	$err(x, y)$
0	100.00000	10.0000000	10.0000000	-10099.0	101989801.00
1	0.9731283	−0.0441109	0.0014698	0.05224	0.0027292029
2	0.9814327	−0.0431786	0.0002079	0.03604	0.0012991802
3	0.9884544	−0.0423522	−0.0008708	0.02224	0.0004946186
4	0.9933328	−0.0417422	−0.0016254	0.01259	0.0001585484
5	0.9962779	−0.0414043	−0.0020860	0.00674	0.0000454583
6	0.9979090	−0.0412176	−0.0023421	0.00349	0.0000122179
7	0.9987692	−0.0411196	−0.0024775	0.00178	0.0000031729
8	0.9992112	−0.0410697	−0.0025472	0.00090	0.0000008144
9	0.9994352	−0.0410446	−0.0025884	0.00045	0.0000002078
10	0.9995481	−0.0410322	−0.0026003	0.00023	0.0000000552

Table 4. Result for Example 2: parameters are chosen as (18)

Iteration k	$x_1 = g_1(x)$	$x_2 = g_2(x)$	$x_3 = g_3(x)$	$g_4(x)$	$err(x, y)$
0	0.0000000	12.000000	10.000000	12.000000	3.0000000000
1	9.9648510	0.0482032	0.0038433	0.0168976	0.0051416657
2	9.9820220	0.0244470	0.0019224	0.0083914	0.0012990283
3	9.9909069	0.0123120	0.0009614	0.0041812	0.0003265513
4	9.9954270	0.0061792	0.0004807	0.0020870	0.0000818680
5	9.9977069	0.0030953	0.0002404	0.0010426	0.0000207149
6	9.9988517	0.0015491	0.0001202	0.0005211	0.0000051277
7	9.9994255	0.0007749	0.0000601	0.0002605	0.0000022548
8	9.9997126	0.0003875	0.0000300	0.0001302	0.0000004149
9	9.9998563	0.0001938	0.0000150	0.0000651	0.0000000949

Table 5. Result for Example 2: parameters are chosen as (19)

Iteration k	$x_1 = g_1(x)$	$x_2 = g_2(x)$	$x_3 = g_3(x)$	$g_4(x)$	$err(x, y)$
0	0.0000000	12.000000	10.000000	12.000000	3.000000000
1	0.1044820	4.4236752	−2.0922554	−7.5641480	61.61724201
2	0.6171098	6.4273221	−3.0063636	−5.9619322	44.70987272
3	9.9909068	0.0123130	0.0009614	0.0041812	0.000326566
4	1.9764287	7.4448462	−3.5658166	−4.1445417	29.89334052
5	9.9977068	0.0030954	0.0002404	0.0010426	0.000020496
6	4.2760527	6.5637328	−3.3723907	−2.5326051	17.78712612
7	9.9994255	0.0007749	0.0000601	0.0002605	0.000001433
8	6.9098232	4.2557367	−2.5625675	−1.397075	8.518383209
9	9.9998563	0.0001938	0.0000150	0.0000651	0.000000090

Table 6. Result for Example 2: parameters are chosen as (20)

Iteration k	$x_1 = g_1(x)$	$x_2 = g_2(x)$	$x_3 = g_3(x)$	$g_4(x)$	$err(x, y)$
0	0.0000000	12.000000	10.000000	12.000000	3.000000000
1	0.1044820	4.4236272	−2.092255	−7.564148	61.61724201
2	0.6171093	6.4273221	−3.006363	−5.961932	44.70987272
3	1.9798724	7.4434382	−3.567583	−4.144273	29.90688757
4	4.2775707	6.5637865	−3.372899	−2.531542	17.78546108
5	6.9101411	4.2561976	−2.562598	−1.396260	8.516529467
6	9.1255582	1.7914574	−1.643263	−0.726247	3.227828695
7	10.5546940	0.0222829	−0.941123	−0.364146	1.018812842
8	10.3568929	−0.0183657	−0.490754	−0.152227	0.264350720
9	10.1964967	−0.0142579	−0.249846	−0.067607	0.067197271
10	10.1028196	−0.0084258	−0.125962	−0.031568	0.016934213
11	10.0525592	−0.0045411	−0.063230	−0.015212	0.004250175
12	10.0265677	−0.0023531	−0.031676	−0.007461	0.001064606
13	10.0133560	−0.0011972	−0.015853	−0.001838	0.000066634
14	10.0033525	−0.0003032	−0.003966	−0.000916	0.000016662
15	10.0016774	−0.0001519	−0.001983	−0.000457	0.000004166
16	10.0008393	−0.0000760	−0.000991	−0.000228	0.000001014
17	10.0004196	−0.0000380	−0.000495	−0.000057	0.000000065

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